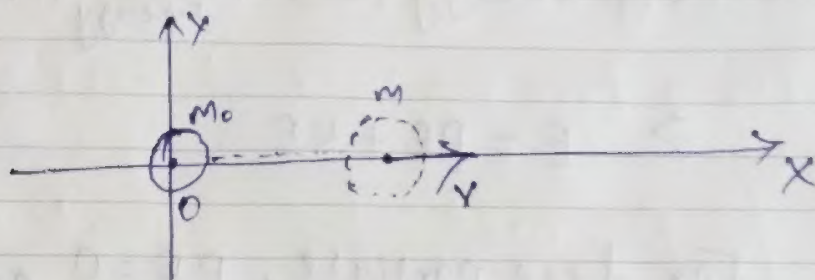


Derivation of $E=mc^2$



In Einstein's Special theory of relativity,
When any object is moving with velocity v
Then its relativistic mass is given by

$$m = \gamma m_0$$

Where γ is Lorentz Factor

and m_0 is rest mass of the object.

According to Einstein's Special theory of relativity and Lorentz Transformation,

$$\text{Lorentz Factor } (\gamma) = \frac{1}{\sqrt{1 - v^2/c^2}}$$

and Relativistic momentum $p = \gamma m_0 v = mv$

From Mechanics, we know that

$$\text{Total Energy (E)} = \underset{\text{Energy}}{\text{Potential (PE)}} + \underset{\text{Energy}}{\text{Kinetic (KE)}}$$

$$\Rightarrow E = PE + KE \quad \text{---(i)}$$

For free particle, $PE = 0$ and $KE = \frac{1}{2}mv^2$

$$\Rightarrow E = 0 + \frac{1}{2}mv^2 = \frac{1}{2}mv^2 \quad \text{---(ii)}$$

According to Work-Energy Theorem,
we know,

$$KE = \text{work done}$$

$$= \int F \, dx = \int \frac{dP}{dt} \, dx$$

$$= \int \frac{d(mv)}{dt} \, dx = \int \left(v \frac{dm}{dt} + m \frac{dv}{dt} \right) dx$$

$$= \int v \frac{dm}{dt} \, dx + m \frac{dv}{dt} \, dx = \int v \, dm \frac{dx}{dt} + m \, dv \frac{dx}{dt}$$

$$KE = \int v^2 dm + m v dv \quad \text{---(iii)}$$

$$m = \gamma m_0$$

$$\Rightarrow m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow m^2 \left(1 - \frac{v^2}{c^2}\right) = m_0^2$$

$$\Rightarrow m^2 (c^2 - v^2) = m_0^2 c^2$$

$$\Rightarrow m^2 c^2 - m^2 v^2 = m_0^2 c^2$$

change into differential form

$$\Rightarrow c^2 2m dm - 2v^2 m dm - 2m^2 v dv = 0$$

$$\Rightarrow c^2 dm = v^2 dm + m v dv \quad \text{---(iv)}$$

From eqn (iii) and (iv)

$$KE = \int_{m_0}^m c^2 dm = c^2 (m - m_0)$$

$$\Rightarrow \boxed{KE = (m - m_0) c^2} \quad \text{---(v)}$$

Here $m - m_0 = \text{change in mass} = \Delta m$

$$\Rightarrow E = (m - m_0) c^2 \quad \text{---(vi)}$$

Here Δm implies increase in Inertial of the object.

Mass is the measure of Inertia.

If we generalise Δm by m , then

For a free particle of mass m ,

$$\boxed{E = mc^2} \quad \text{--- (vii)}$$

This is known as principle of Mass-Energy Equivalence.

This principle was proposed by Einstein in his original paper

"Does the Inertia of a body depend upon its energy contents?"

on

21 November 1905